2019-Sep-17 (pre-lecture) — Prof. Yun William Yu

Tuesday, September 17, 2019 2:29 AM

MAT B44 - 2019

Outline:

- Exact differentials
- Integrating Factors
- Checking solutions
- Bernoulli Equation
- Qualitative Methods

Last time:

We defined an exact differential $P(x,y)\,dx + Q(x,y)\,dy = 0$ if $\exists\, f(x,y)$ s.t. $P(x,y) = \frac{\partial}{\partial x} f(x,y)$ and $Q(x,y) = \frac{\partial}{\partial y} f(x,y)$.

We can recognize it by $\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial x} Q(x,y)$.

Recognizing an exact differential is easy. But can we solve for $f(x,y)$?

Yes. (Tenenbaum 9.43) Given $P(x,y)\,dx + Q(x,y)\,dy = 0$ an exact differential,

$$f(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy, \quad \text{where}$$

the line segments $(x_0,y_0) — (x,y_0)$ and $(x_0,y_0) — (x_0,y)$ lie entirely in R.

Ex. $\left(\frac{2xy+1}{y}\right)dx + \left(\frac{y-x}{y^2}\right)dy = 0$

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{2xy+1}{y}\right) &= \frac{\partial}{\partial y}\left(2x + \frac{1}{y}\right) = -\frac{1}{y^2} \\ \frac{\partial}{\partial x}\left(\frac{y-x}{y^2}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{y^2}\right) = -\frac{1}{y^2} \end{aligned}\right\} \text{exact}$$

So long as $y > 0$, all the partials exist & are continuous.

We choose $x_0 = 0$, $y_0 = 1$.

$$f(x,y) = \int_0^x \frac{2\bar{x}y+1}{y}\,d\bar{x} + \int_1^y \frac{\bar{y}-0}{\bar{y}^2}\,dy$$

$$= \int^x 2\bar{x}\,d\bar{x} + \int^x \frac{1}{\ }\,d\bar{x} + \int^y \frac{1}{\bar{y}}\,d\bar{y}$$

$$= \int_0^x 2\bar{x}\, d\bar{x} + \int_0^x \frac{1}{y} d\bar{x} + \int_1^y \frac{1}{\bar{y}} d\bar{y}$$

$$= x^2 + \frac{x}{y} + \ln|y|$$

Thus $f(x,y) = x^2 + \frac{x}{y} + \ln|y| = c$ solves the ODE.

Alternately, can try solving for $f(x,y)$ (heuristic)

$$(3e^{3x} y - 2x)\,dx + e^{3x}\,dy = 0$$

$$\int (3e^{3x} y - 2x)\,dx = e^{3x} y - x^2 + C_1(y)$$

$$\int e^{3x}\,dy = e^{3x} y + C_2(x)$$

$$\Rightarrow f(x,y) = e^{3x} y - x^2.$$ ← double check

So $e^{3x} y - x^2 = C.$

Also, can sometimes look up common integrable combinations in Tenenbaum, Table, Lesson 10A.

Integrating Factors

Sometimes, given an inexact $P(x,y)\,dx + Q(x,y)\,dy = 0$, we can convert it to an exact one by multiplying.

Define A multiplying factor which will convert an inexact ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$ into an exact one $hP(x,y)\,dx + hQ(x,y)\,dy = 0$ is called an Integrating Factor

Ex. $(y^2+y)\,dx - x\,dy = 0$

Not exact because $\frac{\partial}{\partial y}(y^2+y) = 2y+1$, $\frac{\partial}{\partial x}(-x) = -1$.

However, if we multiply by y^{-2}, ← integrating factor

$$y^{-2}(y^2+y)\,dx - y^{-2} x\,dy = 0$$

$$\Rightarrow (1+y^{-1})\,dx - xy^{-2}\,dy = 0$$

$$\frac{\partial}{\partial y}(1+y^{-1}) = -\frac{1}{y^2} \quad , \quad \frac{\partial}{\partial x}(-xy^{-2}) = -\frac{1}{y^2}. \quad \leftarrow \text{exact}$$

Mentimeter: find an IF for

$$xy\,dx + (1+x^2)\,dy = 0$$

$$(x^2+y^2+x)\,dx + xy\,dy = 0$$

Integrating factors are generally quite hard to find, so we will not spend that much time on them, but you should be aware that a few common types of ODEs have known IFs.

Linear ODE first order (Lesson 11B)

Given $\frac{dy}{dx} + P(x)y = Q(x)$, a linear differential eqn of first order,

a known IF is $e^{\int P(x)\,dx}$.

Ex. $y' + y\cos x = \frac{1}{2}\sin 2x$

$$\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$$

Let $IF = e^{\int \cos x\,dx} = e^{\sin x}$.

$$\frac{dy}{dx}e^{\sin x} + y\cos x\, e^{\sin x} = \frac{1}{2}\sin 2x\, e^{\sin x}$$

$$dy\, e^{\sin x} + dx\left(y\cos x\, e^{\sin x} - \frac{1}{2}\sin 2x\, e^{\sin x}\right) = 0$$

$$dx\, e^{\sin x}\left(y\cos x - \frac{1}{2}\sin 2x\right) + dy\, e^{\sin x} = 0$$

Is this exact?

$$\frac{\partial}{\partial y}\left(y\cos x - \frac{1}{2}\sin 2x\right)e^{\sin x} = (\cos x)\, e^{\sin x}$$

$$\frac{\partial}{\partial x}\left(e^{\sin x}\right) = (\cos x)\, e^{\sin x} \quad // \quad \text{Yes.}$$

What is $f(x,y)$?

Recall $f(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)$

or

$$f(x,y) = \int_{x_0}^{x} P(x,y_0)\,dx + \int_{y_0}^{y} Q(x,y)$$

Let $x_0 = 0$, $y_0 = 0$

$$f(x,y) = \int_0^x \left(0\cos\bar{x} - \tfrac{1}{2}\sin 2\bar{x}\right) e^{\sin\bar{x}}\,d\bar{x} + \int_0^y e^{\sin x}\,d\bar{y}$$

$$= \int_0^x -\tfrac{1}{2}\sin 2\bar{x}\, e^{\sin\bar{x}}\,dx + ye^{\sin x}$$

$$= -\int_0^x \sin\bar{x}\cos\bar{x}\, e^{\sin\bar{x}}\,dx + ye^{\sin x}$$

Let $u = \sin\bar{x}$, $du = \cos\bar{x}\,d\bar{x}$

$$= -\int_0^{\sin x} u e^u\,du + ye^{\sin x}$$

$$= -e^u(u-1)\Big|_0^{\sin x} + ye^{\sin x}$$

$$f(x,y) = -e^{\sin x}(\sin x - 1) - 1 + ye^{\sin x} = C$$

$$ye^{\sin x} = C + e^{\sin x}(\sin x - 1) + 1$$

$$y = Ce^{-\sin x} + \sin x - 1 + e^{-\sin x}$$

$$y = (C+1)e^{-\sin x} + \sin x - 1$$

$$y = Ce^{-\sin x} + \sin x - 1$$

Checking solutions: $y' + y\cos x = \frac{1}{2}\sin 2x$

Is $y = 5e^{-\sin x} + \sin x - 1$ a solution?

$$y' = -5\cos x\, e^{-\sin x} + \cos x$$

$$-5\cos x\, e^{-\sin x} + \cos x + \left(5e^{-\sin x} + \sin x - 1\right)\cos x = \frac{1}{2}\sin 2x$$

$$\sin x\cos x = \sin x\cos x. \quad \checkmark$$

Going backwards from a solution:

Ex. $y = Ce^{-\sin x} + \sin x - 1$, a 1-parameter family of solutions.

Differentiating, $y' = -C\cos x\, e^{-\sin x} + \cos x$ is clearly a solution, but contains an arbitrary constant term.

Let's get rid of it.

Let's get rid of it.

$$y\cos x + y' = C\cos x\, e^{-\sin x} + \sin x\cos x - \cos x$$
$$\qquad - C\cos x\, e^{-\sin x} + \cos x$$

$$y\cos x + y' = \sin x\cos x$$

Ex Sometimes, we can use implicit differentiation, to directly get rid of the constant

Given sol. $x^2+y^2=C$, for some constant C.

$$2x+2yy'=0.$$

$$x+yy'=0. \quad \leftarrow \text{ODE with } x^2+y^2=C \text{ as family of solutions}$$

Bernoulli Equation

A special type of first-order ODE is the Bernoulli Differential Equation (not to be confused with the Bernoulli's Principle about fluid flow) given by $\frac{dy}{dx}+P(x)y=Q(x)y^n$.

To solve it, we can first multiply by $(1-n)y^{-n}$

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx}+(1-n)P(x)(y^{1-n}) = (1-n)Q(x)$$

Substitute $u=y^{1-n}$, $du=(1-n)y^{-n}dy$

$$\Rightarrow \frac{du}{dx}+(1-n)P(x)\cdot u = (1-n)Q(x)$$

This is now a linear first-order ODE, so can use integrating factor $e^{\int(1-n)P(x)dx}$

Ex. $y'+xy=\frac{x}{y^3}$, $y\neq 0$.

multiply by $4y^3 \Rightarrow 4y^3y'+4xy^4=4x$

substitute $u=y^4$, $du=4y^3dy \Rightarrow u'+4ux=4x$

$IF = e^{\int 4x\,dx}=e^{2x^2} \Rightarrow e^{2x^2}u'+4e^{2x^2}(ux-x)=0$

exact differential $\Rightarrow e^{2x^2}du+4e^{2x^2}(ux-x)dx=0$

$$\Rightarrow ue^{2x^2}-e^{2x^2}=C$$
$$ue^{2x^2}=C+e^{2x^2}$$
$$u=Ce^{-2x^2}+1$$
$$y^4=Ce^{-2x^2}+1$$

Qualitative methods:

$$y^4 = Ce^{-4x} + 1$$

Qualitative methods:

Autonomous

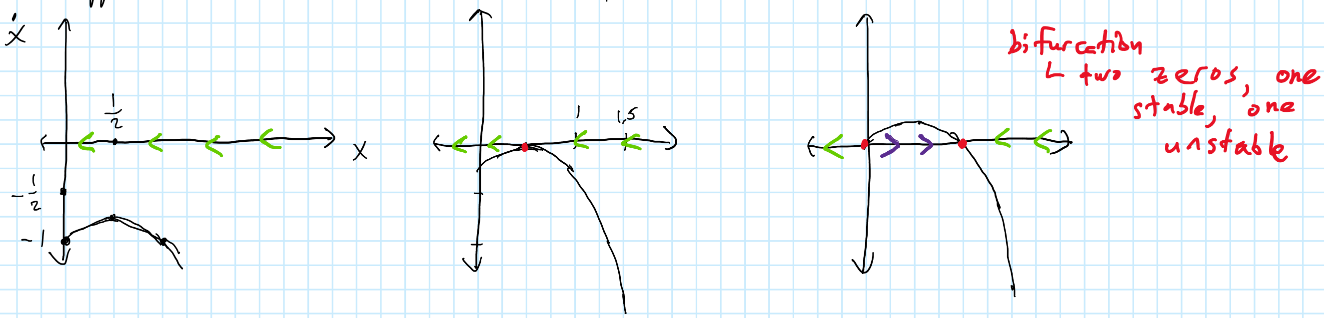
Consider the logistic growth model

(Teschl 1.15 Notice, of Bernoulli type, and also separable.)

$$\dot{x} = (1-x)x - h$$

$$\dot{x} = -x^2 + x - h = -\left(x - \tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4} - h\right)$$

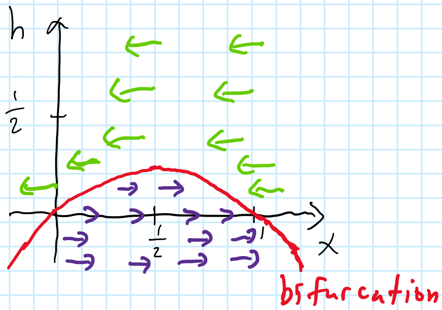
Suppose $h = 1$ $\qquad h = \frac{1}{4}$ $\qquad h = 0$



Let's plot the zeros of x as a function of h.

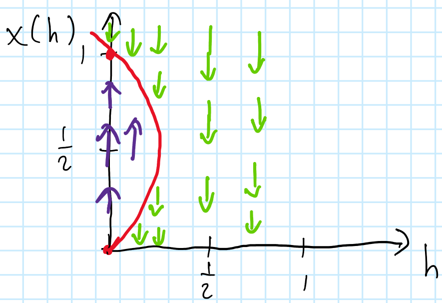
$$(1-x)x - h = 0$$

$$h = (1-x)x$$



When $h > \frac{1}{4}$, there are no zeros, and $\dot{x} < 0$, so x always goes down

When $h < \frac{1}{4}$, there are two zeros. $\dot{x} > 0$ in between, so x goes up but down everywhere else.

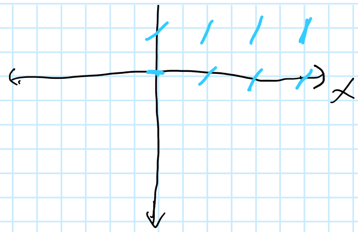


Note, some texts like to flip the axis.

Direction Fields

$y' = x + y$ (or $\dot{x}(t) = x + t$)

We can plot the slope at every point, but this is slow.

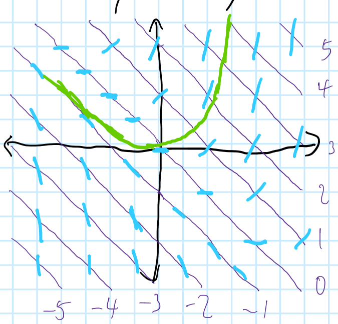


every point, but this is slow.

Def. An isocline as a curve where $y' = c$, for c a constant.

This makes plotting by hand much easier.

$y' = x + y$



$0 = x + y$ $\quad$ $1 = x + y$ $\quad$ $c = x + y$

$y = -x$ $\quad$ $y = -x + 1$ $\quad$ $y = -x + c$

Integral curves can be plotted tangent to each of the slope lines

$y = e^x - x - 1$

Can also plot via Matlab, Wolfram Alpha, etc.

WolframAlpha $\to$ slope field $\to$ $\{1, x+y\}$

or $\left\{\frac{1}{\sqrt{1+(x+y)^2}}, \frac{x+y}{\sqrt{1+(x+y)^2}}\right\}$

WolframAlpha computational intelligence.

slope field

Extended Keyboard Upload Examples Random

Assuming "slope field" refers to a computation | Use as referring to a mathematical definition instead

Computational Inputs:

» vector field: {0.5/sqrt(1+(x+y)^2),0.5*(x+

» variable 1: x

» lower limit 1: -5

» upper limit 1: 5

» variable 2: y

» lower limit 2: -5

» upper limit 2: 5

Compute